STRUCTURAL INVARIANCE: A LINK BETWEEN CHAOS AND RANDOM MATRICES

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The concept of structural invariance previously introduced by the authors is used to argue that the connection between random matrix theory and quantum systems with a chaotic classical counterpart is in fact largely exact in the semiclassical limit, holding for all correlation functions and all energy ranges. This goes considerably further than the usual results obtained through periodic orbit theory.These results hold for eigenvalues of bounded time-independent systems as well as for eigenphases of periodically kicked systems and scattering systems.

1 Introduction

The purpose of this paper is to establish a firm basis for a well-known association beween random matrix theory on the one hand and quantum systems with a chaotic classical counterpart on the other. The key concept in our approach will be that of structural invariance already introduced in a previous paper [1,2]. Since this concept is at the very heart of the issue, we start by describing it in the most general terms possible. We then return to our specific purpose later.

The point at issue is the construction of a reasonable ensemble given an individual system. Such problems are not really new to physics: indeed, whenever one attempts to describe an individual system by an ensemble, as occurs for example in statistical mechanics, the problem of deriving the ensemble from the individual system arises. In such cases it is well known that one first needs to identify a set of relevant properties. Once this has been done, we define the structural invariance group of the object as the group of all transformations which transform the given object into one with the same relevant properties. To fix ideas, let us consider first the following example: Take an arbitrary binary sequence of N elements. Assume first that this sequence has no specific properties that "strike the eye". The structural invariance group then consists of all maps which carry binary sequences into other binary sequences, without any restrictions on these maps. On the other hand, assume that the sequence contains, say, seventy percent ones. This cannot be chance, at least for $N \gg 1$. We therefore must limit the structural invariance group to those transformations which respect that property. The structural invariance group is therefore the group of all permutations acting on the binary sequence. If further, say, the arrangement is periodic with period r, then the structural invariance group will

only consist of those permutations which respect that feature and will therefore be isomorphic to the symmetric group over r elements. Note carefully how the structural invariance group becomes *smaller* as the symmetry of the object increases. Indeed, it may well be that the structural invariance group is in a technical sense complementary [6,2] to the symmetry group with respect to the largest group of all possible transformations. As one readily sees in the above example, the concept of structural invariance group is somewhat ambiguous in pure theory, yet in practice the assignment of a given group to an object hardly ever presents real problems.

Once the structural invariance group G of an object has been defined, there is usually no difficulty in embedding that object in a "natural" ensemble. The procedure is as follows: let the group G act on the original object. In this way it defines a set Σ, on which an action of G is defined. If G has further an invariant measure, then this measure induces a measure on Σ, so that we have indeed constructed an ensemble of which the original object is a typical element. If we now study such properties of the ensemble as are *ergodic*, that is, which hold with probability one for any given element of the ensemble, we can reasonably conclude that the original object must have these properties. If it does not, this strongly indicates that the structural invariance group was incorrectly identified.

While these concepts may well be applicable in many different fields, the authors were interested in applying them to the following example: ever since the pioneering papers of Berry [3] the connection between the spectral properties of a quantum system having a chaotic classical counterpart with those of random matrix theory (RMT) has been a subject of intense research. The result of many independent numerical tests has been the following: as long as no symmetries are present (in particular no discrete symmetries), the spectra of the resulting quantum system has the behaviour of the appropiate matrix ensemble [5,7]. This means that it belongs to the universality class of the so-called Gaussian Orthogonal Ensemble (GOE) if it is invariant under time reversal and to that of the Gaussian Unitary Ensemble (GUE) otherwise. On the other hand, if symmetries do exist, they are taken into account by introducing independent ensembles for each symmetry sector. The validity of such modelling has been supported by a large quantity of numerical work as well as by semiclassical considerations based on the Gutzwiller sum formula [9,4]. Some apparent exceptions to this association between chaos and random matrix behaviour are known (e.g. the modular billiard on the surface of constant negative curvature), but these have been found to be due to peculiar hidden discrete symmetries of number-theoretical origin [10].

In this paper, we shall explain this association in terms of the concept of

structural invariance outlined above. Such an approach cannot, by itself alone, rigorously prove anything for a specific system. Nevertheless, it provides a powerful instrument for justifying the association of a given type of ensemble to a given class of systems. Further, as we shall see, it provides results that periodic orbit theory is (up to now) quite unable to show: in particular, it can be shown that the spectral properties predicted by RMT hold exactly, not only for large energy differences, as in the periodic orbit approach. It can also be seen that arbitrary correlation functions are correctly given by RMT, whereas this is a highly non-trivial problem in periodic orbit theory. Thus, it can be said that the statements we make are far stronger than those which can be shown by other means, but that we are limited to saying that they are true "generically" or "with probability one". Even this, however, may not be as severe a drawback as might at first appear: Indeed, assume one were able rigorously to show that any chaotic system without discrete symmetries had the appropiate RMT behaviour. Since proving the absence of *any* kind of symmetry whatsoever is an extremely difficult task, the usefulness of even such a strong theorem is very limited indeed. Therefore, precisely from the very standpoint of mathematical rigor, such a theorem would be useless. Of course, combined with physical intuition as to the "likelihood" of hidden symmetries, it becomes useful again, but at that level our arguments are, we believe, equally compelling.

The rest of this paper is organized as follows: in Sec. 2 we discuss the application of the structural invariance concepts described above to the case of a canonical map without any structure. From this we obtain a well-known connection between the eigenphases of a chaotic periodically time-dependent system and the so-called Circular Unitary Ensemble defined by Cartan. In Sec. 3 we show how various symmetries of the problem reduce the structural invariance group and we show that the ensembles one is led to consider are exactly those which are usually associated with such systems. In Sec. 4 we show how the results for eigenphases of maps generalize to the case of Hamiltonian systems and their eigenvalues. In Sec. 5 we present some comments and conclusions.

2 Structural invariance for canonical maps

In the following, we shall consider bijective canonical maps C from a compact phase space Γ onto itself. This might represent a number of things, such as the Poincaré map of a bounded time-independent system, a scattering map or the time evolution over one period of a periodically driven time-dependent system. For simplicity we shall usually think of it as the last of these.

We shall take the point of view that the relevant properties of a canonical map are those which give rise to recognizable organized structures upon iterating the map. This is rather natural in view of the examples given above. Thus we are led to consider invariant tori and cantori as part of the properties which the structural invariance group must leave invariant. Further, such properties as the existence of discrete symmetries must of course also be preserved. On the other hand the exact location and properties of the isolated unstable periodic orbits of the system are of course not relevant. Of further relevance, though rather less trivial, is time-reversal invariance. Indeed, we say that a map C is time-reversal invariant (TRI) if there exists a non canonical map T with T^2 being the identity, such that $CT = TC^{-1}$. The classical example of this, of course, is the usual case where T only changes the sign of the momenta but other cases are known as well.

Let us now consider a map C which is wholly structureless. Then, defining the group of all bijective canonical transformations to be $\mathcal{G}$, one finds that the structural invariance group of C is $\mathcal{G} \times \mathcal{G}$ with the following action:

$$(S, S') : C \longrightarrow SCS'. \tag{1}$$

The set Σ is then given as the set of all canonical transformations, which is itself the group $\mathcal{G}$, and the action of $\mathcal{G} \times \mathcal{G}$ on $\mathcal{G}$ is given by Eq. (1). The example therefore appears as an entirely trivial one.

In making Σ to an ensemble, however, we encounter a fundamental problem: The groups $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$ are both infinite-dimensional and do not have a known, useful invariant measure. There is therefore no natural way to define a measure on Σ dictated by invariance considerations alone.

On the other hand, we are interested in the consequences of the chaotic nature of the map for quantum mechanics. In this case, a chaotic transformation induces (via any of a large number of quantization procedures) a unitary map from a Hilbert space onto itself. Since we are considering a compact phase space, this Hilbert space is essentially finite dimensional with dimension N given by $|\Gamma|/(2\pi\hbar)^f$, where $|\Gamma|$ is the volume of Γ and f is the number of degrees of freedom in the system. If we are considering a Poincaré map, for example, then N as well as $|\Gamma|$ are related to the energy scale. Quite generally, it can be argued that the semiclassical limit, which is the only case we shall be concerned with, corresponds to $N \gg 1$. In this case we can disregard the complex issues concerning the ambiguities involved in quantization and assign to every canonical transformation a unitary transformation as shown by Dirac[11] [8]. Again up to effects which are outside of the semiclassical approximation, this unitary map can be viewed as a map on an N-dimensional space. That is, in the final analysis, we have mapped the group $\mathcal{G}$ onto $U(N)$ within the

appropiate approximations.

From this, however, our conclusion follows immediately: indeed, there is a Haar measure on $U(N)$ which is invariant under the left and right action of $U(N)$ on itself. We therefore reach the following conclusion: a completely structureless map is to be associated with the ensemble of all unitary matrices endowed with the corresponding Haar measure. This corresponds to saying that the matrices are generic elements of the so-called Circular Unitary Ensemble (CUE). This conclusion had been reached by entirely different means for the specific cases of the scattering matrix and of periodically driven systems. These methods employed semiclassical techniques based on periodic orbit theory, which could only show that the two-point function of the eigenphases corresponds with the CUE result for large eigenphase differences. Our result implies much more: it claims the same result for all energy ranges and for all properties of the eigenphases. It also extends to eigenfunctions without any problems, since our claim holds for the unitary matrix representing the canonical map C. On the other hand, as we said in the introduction, it is not capable of showing that a specific system has these properties with absolute certainty. Rather, it very strongly suggests that if the properties of the CUE are found to be violated, some significant structure must be present in the map, which has been overlooked.

3 Implications of various structural properties

In the following we discuss maps which are not wholly structureless and show that they do indeed give rise to the ensembles by which they are successfully described in RMT. Let us start by the most important case of time-reversal invariance. We need to find the subgroup $\mathcal{H}$ of $\mathcal{G} \times \mathcal{G}$ which leaves the TRI property unaltered. An easy calculation shows that [1]

$$\mathcal{H} = \{(S, S') : S' = TS^{-1}T\}. \tag{2}$$

From this follows the corresponding quantum-mechanical result. Let us call U_C the unitary matrix corresponding to the canonical transformation C. Then the transformations that correspond to those that are induced by $\mathcal{H}$ are the following:

$$U_S : C \longrightarrow U_S U_C U_T U_S^{-1} U_T, \tag{3}$$

where U_T represents the *anti-unitary* matrix associated to T. If T can indeed be represented by complex conjugation (as we shall assume henceforth, the opposite case being the one that gives rise to the symplectic ensembles), then the action of U_S on the *unitary symmetric* matrix U_C is given by $U_S U_C U_S^t$,

where ^{t} denotes the transpose. The set Σ is now the set of all symmetric unitary matrices and the measure $d\mu$ is the one that remains invariant under the action defined by Eq. (3). Again by a standard theorem, this measure is unique and given by the so-called Circular Orthogonal Ensemble (COE).

Let us now assume that the map C has a given symmetry, say P. This means that $PC = CP$. Under these circumstances, the subgroup of $\mathcal{G} \times \mathcal{G}$ which leaves the symmetry invariant is given by

$$\mathcal{H} = \{(S, S') : SP = PS; S'P = PS'\}. \tag{4}$$

Again, the set Σ consists of all maps C having the symmetry P. As a subset of $U(N)$, it turns out to be the direct sum of all the various symmetry sectors of P. If P has non-trivial representations, then degeneracy follows trivially, whereas the independence of distinct symmetry sectors is equally clear. None of these results can be easily derived in such generality from periodic orbit theory. Also, we see the reason why time-reversal invariance plays a role distinct from other Abelian symmetries such as parity. Again this is far from obvious in a treatment based on periodic orbit theory, since in both cases one has systematically doubly degenerate orbits.

Let us now consider a somewhat more involved case: Let us aasume that we have a system with time-reversal invariance *and* a discrete symmetry of order higher than two. If we reduce the classical phase space according to symmetry sectors, two cases can present themselves: First, all symmetry sectors are separately TRI. In this case (which has been the most common so far) the map can be described as an uncorrelated superposition of COE's. On the other hand, it is equally possible that time-reversal carries one of the symmetry sectors into the other and vice versa. In this case it is easy to see that T must be represented by an antiunitary matrix which interchanges the two sectors. This leads to a structural invariance group with the following action on the two blocks

$$(U_1, U_2) : (U_{C,1}, U_{C,2}) \longrightarrow (U_1 U_{C,1} U_2^t, U_2 U_{C,2} U_1^t), \tag{5}$$

where the indices denote the symmetry sector on which the corresponding matrices operate. From this it follows that the two sectors have degenerate eigenphases (Kramer's degeneracy) and that these eigenphases obey CUE statistics.

4 From eigenphases to eigenvalues

So far we have only treated the case of a canonical map, the iteration of which gives the time evolution. An important question, however, is to transfer this

analysis to flows generated by a time-independent Hamiltonian. The immediate problem with this is that we must find a way to account for the fact that the overall density of states is *not* given by that of the corresponding matrix ensembles and can be rather arbitrary, whereas the fluctuation properties are indeed given by the RMT predictions. This difficulty was absent in the earlier cases since the density of states was correctly predicted to be uniform.

To do this, we consider the energy-dependent Poincaré map $C_E(p_s, q_s)$, where the subscript s indicates that these refer to phase space variables in the Poincaré surface of section. If one denotes by $T(E)$ the corresponding unitary map, it follows from the above that the eigenphases of $T(E)$ are distributed according to some appropiate random matrix ensemble. We now need to carry this information over to the eigenvalue distribution.

To this end one proceeds as follows: Bogomolny [12] has shown that a semiclassical quantization condition is the following: if E is an eigenvalue, then

$$\det(\mathbf{1} - T(E)) = 0. \tag{6}$$

Thus, if the eigenphases of $T(E)$ are denoted by $\exp(i\phi_j(E))$, then every time a given $\phi_j(E)$ goes through zero, E is an eigenvalue. It turns out that the whole procedure is only semiclassical, as the map $T(E)$ is only unitary in the semiclassical limit, but this is not a problem, since this limit is in any case the only one we are able to handle. Also, outside of the true semiclassical limit, the relation between chaos and RMT is much more subtle: in particular, one has problems such as (transient) Anderson localization, in which chaotic behaviour and randomly distributed eigenvalues coincide. One also then has to deal with finite tunneling probabilities and other phenomena associated with the structure of our canonical map in the complex plane, which we have left out of consideration entirely, as our understanding of these is still very incomplete.

It therefore appears that we have reduced the problem of determining the spectrum of a Hamiltonian $H(p, q)$ to the study of the energy dependence of the eigenphases of the quantized version of its Poincaré map. To handle this problem, we must first know how the Poincaré map changes under infinitesimal changes of E. To this end, let us consider $S(E; q_s, q_s')$, defined as the action along the classical path connecting the two surface points at energy E. It follows from standard arguments that this is in fact the generating function of $C_E(p_s, q_s)$. From this one obtains after some manipulation

$$C_{E+\Delta E} C_E^{-1} = \Phi_{\Delta E}, \tag{7}$$

where $\Phi_{\Delta E}$ is defined as follows: consider the "Hamiltonian" $\mathcal{T}_E(p_s, q_s)$, which is defined as the time necessary to return to the Poincaré surface if one starts

from (p_s, q_s). This "Hamiltonian" generates a flow on the Poincaré surface and $\Phi_{\Delta E}$ is the infinitesimal canonical transformation corresponding to following this flow for a "time" ΔE.

If we now follow this through the quantization procedure, we obtain the following: let the eigenphases $\phi_j(E)$ be defined as previously. Let $\psi_j(E)$ be the corresponding eigenfunctions. If we now denote by $\mathcal{H}_E$ the self-adjoint operator corresponding to $\mathcal{T}_E$, we finally obtain

$$\frac{d\phi_j}{dE} = \langle\psi_j(E)|\mathcal{H}_E|\psi_j(E)\rangle. \tag{8}$$

Now we must make some key approximations: First, we remmember that we are in the semiclassical limit, that is, that the classical function $\mathcal{T}_E(p_s, q_s)$ is smooth compared with the $\psi_j(E)$. This implies that instead of using, say, the Wigner distribution in computing the r.h.s of Eq. (8), we can use a smoothed version such as the Husimi distribution without great error. Since the $\psi_j(E)$ are eigenfunctions of a matrix representing a totally structureless map, their Husimi distributions will be spread uniformly all over phase space. This would follow from our considerations on structural invariance, but is equally confirmed by a rigorous theorem of Shnirelman's concerning eigenfunctions of chaotic billiards [13]. From this one finally gets

$$\frac{d\phi_j}{dE} = \overline{\mathcal{T}_E(p,q)}, \tag{9}$$

where the overline denotes average over phase space. The crucial points to note are the following: First, the velocity at which the eigenphases move is, in a first approximation, independent of j. This is because there are N eigenphases on the unit circle. Since they move with a velocity on the order of one, they will cross zero at energies which differ by an order of $1/N$, which is natural, since we have chosen our scale of energies to be the classical one. Therefore, from one eigenvalue to the next, the velocity at which the eigenphases moves hardly changes. This means that the CUE (or other RMT) properties of the eigenphases translate directly into corresponding properties of the eigenvalues of the system. Second, however, and equally important: this constancy does not hold forever. Two effects will eventually alter this state of affairs: first, the average on the l.h.s. of Eq. (9) will experience a secular change as E changes. This corresponds to the secular change in the density of states which is usually eliminated by unfolding the spectrum. On the other hand, another effect may well come into play even before this secular change becomes noticeable. The point is that Eq. (9) is only true as a statistical statement and there are fluctuations around the mean velocity. The most obvious cause for fluctuations

are departures of the Husimi distribution from equidistribution. Such deviations are well-known to exist, namely the so-called "scars" near short periodic orbits. It could therefore well be that these accumulated fluctuations could account for some of the effects due to short periodic orbits. To explore this possibility, however, we would presumably require an understanding of scars which we do not have at present. Finally, it should be pointed out that the l.h.s of Eq.(9) is directly related to the phase space volume (see e.g. [12]) and hence to the Weyl formula, so that this formalism recovers well-known results on the overall density of states.

5 Conclusions

We have presented a systematic approach to the connection between RMT and individual dynamical systems. This connection is in a sense of a probabilist type: it rests basically on conclusions of the form: a given system is a typical representative of a certain ensemble, elements of this ensemble have a given property with probability one, therefore the original system has the stated property. While this line of argument is wide open to fundamentalist attacks from the mathematical side, it is undeniably useful at the heuristic level. Further, similar lines of reasoning are frequently used in statistical mechanics when applying an ensemble description to an individual system. A more genuine concern concerns the construction of the ensembles: as we are not able to construct ensembles on the set of all canonical transformations, we must first construct a set of canonical transformations at the purely classical level and then translate this into quantum mechanics in order to obtain a reasonable candidate for a measure. This is undoubtedly unsatisfactory, but it is probably a genuine problem, not merely a measure of the authors' incompetence. In fact, finding a connection between the approach based on periodic orbits and the one we have sketched here is a very interesting open problem. In particular, periodic orbit theory, in order to be consistent witht RMT, strongly suggests that peculiar correlations between periodic orbits must exist for very long orbits[14]. Deriving such results from considerations of structural invariance might indeed be a significant step forward. That this is not trivial, however, depends precisely on the fact that it is not possible to obtain measures on the set of all canonical maps. Similar remarks obtain for quantum localization: as long as the phase space Γ is compact, localization is only transient, and therefore outside the immediate range of application of semiclassics. Non-compact phase spaces, on the other hand, also present problems relating to the existence of an invariant measure.

On the other hand, the method of structural invariance has shown itself

to be a powerful tool: it associates the correct ensemble in every relevant situation. Further, our approach shows that not only do the RMT predictions hold for the two-point function and at large energy distances, but that they should hold at all energy scales and for all correlation functions. This is much more than periodic orbit analysis has achieved yet. In fact the "unreasonable effectiveness" of RMT in describing the quantum analogs of classically chaotic systems has so far eluded any explanation. The result presented here should therefore be viewed as a new and significant one, although entirely expected from the numerical point of view.

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